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Author(s): Otto A. Davis and Andrew Whinston
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EXTERNALITIES, WELFARE, AND THE THEORY OF GAMES[1]

OTTO A. DAVIS AND ANDREW WHINSTON
Carnegie Institute of Technology and Yale University

I. INTRODUCTION

It has traditionally been argued that, if firms create external economies and diseconomies, the proper role of a welfare-maximizing government is to constrain the behavior of firms by arranging rates of taxes and subsidies in order to equate private with social benefit. We attempt to establish both the conditions under which this classical policy prescription might work and is needed, and those under which it cannot be expected to work.

First, we argue that motivation exists for firms themselves to try to eliminate externalities in production through merger. Second, we attempt to show that technological externalities can be divided neatly into two cases, which we label "separable" and "non-separable," respectively. Third, if merger has not eliminated the externalities, we argue that the classical scheme of per unit taxes and subsidies can be clearly successful in equating private with social benefit only in the separable cases. Fourth, if the externality is non-separable, we argue that it is not clear that the classical prescription can work even at the conceptual level, since problems of uncertainty and the non-existence of equilibrium arise. Finally, we note that this latter possibility poses some difficult problems for policy-makers, and we attempt to outline and explore briefly alternative policy approaches.

The analytic approach which we shall employ involves the consideration of two firms in a competitive industry. The traditional or classical approach, on the other hand, often involves an analysis of externality between competitive industries. We choose to depart from this traditional approach for several reasons. First, the firm is an entity which fits more easily into the framework of our analysis. Second, and more fundamental, it is individual decision units—firms—which react to externalities so that it seems more "natural" to conduct the analysis at that level.[2] Furthermore, concentration upon the industry (as opposed to the firm) requires a certain amount of aggregation which tends to mask some of the more important and interesting points at issue. This aggregation is especially misleading with respect to public policy regulation, where the problem is made to appear much more simple than it actually is. Finally, utilization of the firm as the basic analytic unit gives a level of generality

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[2] Interestingly enough, J. de V. Graaff also considers that externalities are a phenomenon which relates to the firm rather than the industry, and, furthermore, he seems to think this point quite important (see his *Theoretical Welfare Economics* [Cambridge: Cambridge University Press, 1957], p. 19).

which is greater than can be obtained through the traditional approach. The reason for this is that no two firms in an industry may be affected identically by an externality, or some firms in an industry may impose (different "amounts" of) an externality upon other production units, while the remaining firms in the industry may not create externalities. And it should be emphasized at the outset that our concern with firms within the same industry is only a device to simplify the analysis. A more elaborate use of subscripts would allow the firms under consideration to be in different industries.

Yet another result of our approach will be a demonstration that externality problems involve many aspects of duopoly problems. This will be particularly striking in the case which we consider—reciprocal externality between two firms—but, peculiarly enough, these duopoly-like problems remain even if the number of firms under consideration is expanded to n. We shall not attempt such an expansion here, however, since all relevant aspects of the problem seem to be contained in the two-firm case, so that a sufficient level of generality can be achieved without resort to additional complications.

II. MOTIVATION AND MERGER

Consider two firms in a purely competitive industry which are related through their cost functions (external economies and diseconomies on the production side).[3] Assume that the cost functions are

$$C_1 = C_1(q_1, q_2)$$
$$C_2 = C_2(q_1, q_2), \tag{1}$$

where the subscripts refer to the respective firms, C represents cost, and q indicates the output level.[4] If each firm maximizes profit, we have the relationships

$$p = \frac{\partial C_1}{\partial q_1} \quad \text{and} \quad p = \frac{\partial C_2}{\partial q_2}, \tag{2}$$

where p represents price.[5] Each firm must maximize its profit with respect to the variable under its control, although the level of its profit depends by assumption upon the output level of the other firm.

It is well known that the welfare associated with the production of the commodity can be measured by the difference between social benefit and social cost, and that in a competitive market the social benefit can be measured by the firms' total revenue, $p(q_1 + q_2)$, while social costs can be measured by the firms' total costs, $C_1(q_1, q_2) + C_2(q_1, q_2)$. It follows that, in order to maximize welfare, the joint profits of the firms must be maximized. In other words, using P to represent profits, let

$$P = P_1 + P_2 = p(q_1 + q_2) - C_1(q_1, q_2) - C_2(q_1, q_2) \tag{3}$$

represent the total profits of these two firms as indicated by the relevant subscripts. A necessary condition for maximization under the indicated assumptions is

$$\frac{\partial P}{\partial q_1} = p - \frac{\partial C_1}{\partial q_1} - \frac{\partial C_2}{\partial q_1} = 0$$
$$\frac{\partial P}{\partial q_2} = p - \frac{\partial C_1}{\partial q_2} - \frac{\partial C_2}{\partial q_2} = 0, \tag{4}$$

[3] The analysis in this paper is conducted within the context of a competitive industry, although some of our results are applicable even if the market structure is not competitive. We have made the competitive assumptions simply because of a desire to make welfare statements which require such a framework.

[4] It should be emphasized that we consider only technological externalities in this paper. We are not concerned with possible problems associated with pecuniary externalities. The usual convexity conditions are assumed whenever appropriate.

[5] Although we assume the two firms to be in the same industry, this assumption is not necessary either here or in the remainder of the paper. All that is required for the general case is to assume two prices, p_1 and p_2, instead of the single price p.

and a sufficient condition is

$$\frac{\partial^2 P}{\partial q_1^2} < 0 , \qquad \frac{\partial^2 P}{\partial q_2^2} < 0$$
$$\frac{\partial^2 P}{\partial q_1^2}\frac{\partial^2 P}{\partial q_2^2} > \left(\frac{\partial^2 P}{\partial q_1 \partial q_2}\right)^2 . \qquad (5)$$

Attention will now be focused on the first-order (necessary) conditions as given in (4). Note that if either $(\partial C_2)/(\partial q_1) \neq 0$ or $(\partial C_1)/(\partial q_2) \neq 0$, then conditions (2) and (4) will not coincide. Due to the technological externalities, profit maximization by the individual firms will not give the greatest social benefit that is possible.[6]

Marshall and Pigou, considering the case of a negatively sloped supply curve, suggested the possible use of taxes and subsidies as one way to handle this type of difficulty. Meade has effected a modern statement of this classical solution.[7] In particular, Meade argues that a tax-subsidy solution is sufficient to achieve the desired welfare-maximizing solution. We shall argue at a later point in this paper that a tax-subsidy approach is not sufficient to guarantee the attainment of a welfare maximum and, furthermore, that in some cases it is not even clear that it will lead to an improvement in social welfare. We shall attempt now to show that this scheme is not necessary for the optimal welfare solution.

In contrast to the above authors, we do not take the firms as given. Rather, we shall argue that there is a "natural unit" for decision-making and that this unit is responsive to market forces. A "natural unit" is defined as one which results after sufficient mergers have taken place to produce a certain "minimal" set of interrelationships with other units in society. In the context of this discussion, these "interrelationships" may be thought of as external economies and diseconomies.

As might be expected, the formation of natural units poses certain problems. These range from the question of how a competitive market structure is maintained in the face of mergers to the question of the terms on which such mergers may be arranged. For the moment, however, let these problems be waived. Then if either or both

$$\frac{\partial C_2}{\partial q_1} \neq 0 , \qquad \frac{\partial C_1}{\partial q_2} \neq 0 \qquad (6)$$

[6] The usual discussion of technological externalities deals in terms of production functions. We work with cost functions merely for convenience. Identical results are achieved when production functions are considered. For example, consider a single firm in a competitive industry, with the production function

$$q_1 = f(L, q_2) , \qquad (a)$$

where q_1 represents the output of firm 1, L represents an input of labor, and q_2 is an output of firm 2 which affects the production of q_1; q_2 is a non-priced and uncontrolled "input" of firm 1. By assumption the firm desires to maximize the following profit function, where P represents profits, p the price of output, and w the wage rate.

$$P = p q_1 - wL . \qquad (b)$$

Note that q_2 does not enter "directly" into (b), but it does affect (b) since we can write

$$P = p f(L, q_2) - wL \qquad (c)$$

by making a simple substitution.

In its attempt to maximize profits the firm would hire labor up to the point where

$$\frac{\partial P}{\partial L} = p \frac{\partial f}{\partial L} - w = 0 . \qquad (d)$$

However, by the externality assumptions, firm 1 does not control the output q_2 of firm 2. Therefore if

$$\frac{\partial P}{\partial q_2} = p \frac{\partial f}{\partial q_2} \neq 0 , \qquad (e)$$

firm 1 would not be able to achieve its over-all profit maximum. This means, under the assumed conditions, that welfare is not maximal.

[7] James E. Meade, "External Economies and Diseconomies in a Competitive Situation," *Economic Journal*, LXII (March, 1952), 54–67. See also chap. x of his *Trade and Welfare* (London: Oxford University Press, 1955).

obtains, it seems clear that either (*a*) there would be some price at which one firm would be willing to acquire the other or (*b*), more generally, gains to both firms can be secured by effecting a merger. These two cases are lumped into one here simply to argue that a tendency toward such mergers is consistent with, if not implied by, the idea of profit maximization. Consequently, there would be a tendency for such mergers to occur, for production externalities to be internalized, and for joint profits (hence social welfare, if competition is maintained) to be maximized. Insofar as this occurs, a natural unit for decision-making would be realized and, assuming competition, welfare maximized without the use of externally imposed subventions or penalties.[8]

We do not claim, of course, that the natural unit will always be achieved. Instead, we are content to point out that motivation exists for the formation of natural units, and we argue that there should be a tendency toward such mergers. However, realism compels us to recognize that such problems as might be associated with decentralized administration within the merged entity might prevent the achievement of the natural unit in some instances. In addition, from the standpoint of social policy, some mergers might result in a change of market structure and, therefore, be deemed socially undesirable. Hence, it seems appropriate to analyze the externality question further in order to determine whether there exist workable schemes for welfare maxima when the natural unit has not been realized.

III. SEPARABILITY AND DOMINANCE

Since we shall argue that the classical tax-subsidy prescription will not achieve the optimal welfare solution in all cases, it seems desirable to try to determine the conditions under which the scheme can or cannot be expected to work. In this regard it is convenient, in order to make clear the distinction between what at a later point we call the "separable" and the "non-separable" types of externalities, to introduce the following definition. A function, $f(x_1, x_2)$, is said to be "separable"[9] if and only if

$$f(x_1, x_2) = f_1(x_1) + f_2(x_2). \quad (7)$$

In other words, separability means that it must be possible to express the function, $f(x_1, x_2)$, as a sum of two functions each of which involves only one variable in its argument.[10]

As a case in point, we may consider a specific example of two interrelated, but separable, cost functions:

$$\begin{aligned} C_1(q_1, q_2) &= A_1 q_1^n + B_1 q_2^m \\ C_2(q_1, q_2) &= A_2 q_2^r - B_2 q_1^s . \end{aligned} \quad (8)$$

The profit maximization condition (2) then gives

$$\begin{aligned} p &= \frac{\partial C_1}{\partial q_1} = n A_1 q_1^{n-1} \\ p &= \frac{\partial C_2}{\partial q_2} = r A_2 q_2^{r-1} . \end{aligned} \quad (9)$$

[8] The idea of natural units is not original with us. George J. Stigler, commenting on the production theory of Alfred Marshall, observed that when production functions are technically related there may exist motivation for combination and merger (*Production and Distribution Theories* [New York: Macmillan Co., 1946], p. 75).

[9] See A. Charnes and C. E. Lemke, "Minimization of Non-linear Separable Convex Functionals," *Naval Research Logistics Quarterly*, Vol. II (June, 1954).

[10] Note that transformations of the kind discussed in A. Charnes and W. W. Cooper, "Non-linear Power of Adjacent Extreme Point Methods in Linear Programming," *Econometrica*, Vol. XXV (January, 1957), are also admitted.

Note in particular that the marginal cost of each firm is given entirely in terms of its own output variable.

The typical cases with which the classical analysis has been concerned have, in fact, assumed the condition of separability.[11] When this assumption is dropped, there is inevitably introduced an element of uncertainty which becomes rather difficult to deal with in terms of traditional tools and concepts. We shall also attempt to show at a later point that the absence of this condition complicates policy choices and, in particular, renders the tax-subsidy scheme practically useless.

For the moment, however, let us continue our analysis under the traditional assumption of separability. Consider the rule "price equals marginal cost" for each firm as represented in (9). Evidently, each firm may calculate its marginal cost unambiguously at every output, and, therefore, it can also determine its output unambiguously in accordance with the stipulated rule.

To bring out the importance of this consequence of separability, it is desirable to reformulate the problem in terms of a two-person non-zero sum continuous game.[12] To accomplish this reformulation we note that by definition

$$P_1 = P_1(q_1, q_2) = pq_1 - C_1(q_1, q_2)$$
$$P_2 = P_2(q_1, q_2) = pq_2 - C_2(q_1, q_2), \quad (10)$$

where p is taken as given by the market. The game aspect of this problem is the fact that the profit level of each firm depends upon the output (strategy) selected by the other firm. Consonant with the assumptions of classical analysis, it is assumed that the game is non-cooperative. Neither firm communicates with or consults the other while making its choice of output. Then the rule of profit maximization gives "price equals marginal cost" for each firm. But since the marginal cost of each firm as stated in (9) is defined entirely in terms of its own output, this rule means that whatever the output chosen by firm 2, there is a unique output which maximizes firm 1's profit. Similarly, for firm 2 there exists a unique output at which, whatever the output of firm 1, its profits are maximal. In the context of game theory, this is the Von Neumann-Morgenstern concept of dominance.

We shall shortly state this result in terms of mathematical theorems and explore more fully its implications within the relatively simpler context of discrete

[11] It may seem presumptuous of us to assert that the typical cases in the classical analysis implicitly have assumed separability. After all, if the discussion is verbal, then it is difficult to determine the implicit assumptions underlying the analysis; and if the discussion is mathematical but utilizes only general functional notation, then implicit assumptions are equally obscure. Both these cases have been the rule in the literature. Witness, for example, P. A. Samuelson, *Foundations of Economic Analysis* (Cambridge, Mass.: Harvard University Press, 1947); W. J. Baumol, *Welfare Economics and the Theory of the State* (Cambridge, Mass.: Harvard University Press, 1952); and J. de V. Graaff, *op. cit.* However, as will become clear from our later discussion, the conclusions of these analyses necessarily require the separability assumption if one presumes that they are correct. However, it is only fair to point out that K. J. Arrow, while discussing consumption externalities, expressed some doubts about the classical tax-subsidy scheme. He observed, "The general feeling is that in these cases, optimal allocation can be achieved by a price system, accompanied by a system of taxes and bounties. However, the problem has been only discussed in simple cases and no system has been shown to have, in the general case, the important property possessed by the price system" ("An Extension of the Basic Theorems of Classical Welfare Economics," *Second Berkeley Symposium on Mathematical Statistics and Probability*, ed. J. Neyman [Berkeley: University of California Press, 1951], pp. 528–29).

[12] For a brief discussion of continuous games, see I. L. Glicksberg, "A Further Generalization of the Kakutani Fixed Point Theorem with Application to Nash Equilibrium Points," *Proceedings of the American Mathematical Society*, III (1952), 170–74.

games. Before leaving the continuous case, however, it seems appropriate to describe what separability means in terms of the familiar graphical approach. In the particular case which we are examining, the total cost of each firm is a function of two variables, its own output and the output of the other firm, and can be represented on a two-dimensional graph by a family of curves relating the firm's costs to its own output for various given levels of the other firm's output. "Separability" means that a change in the other firm's output simply shifts the cost curve of the firm vertically upward or downward.

This case is depicted in Figure 1. Here two cost functions are drawn—$Aq_2^{(1)}$ and $Bq_2^{(2)}$. If the externality is an external economy, then $q_2^{(1)} > q_2^{(2)}$; and if the externality is an external diseconomy, then the inequality is reversed. As we have shown, separability means in the context of this discussion that externalities do not affect marginal cost. Hence, the cost functions differ from each other by the value of a constant (the value of the externality) since the slopes of the tangent lines (marginal cost) of all possible total cost functions at any specified level of output must be equal to each other. Thus, given some specified price and total revenue curve—say Op'—an alteration in the value of the externality, q_2, will not change the optimal output q_1^* of the firm.[13] The effect of separable externalities is strictly intramarginal. They affect the over-all profit position of the firm but do not alter marginal cost and, hence, do not alter the optimal output choice of the firm.

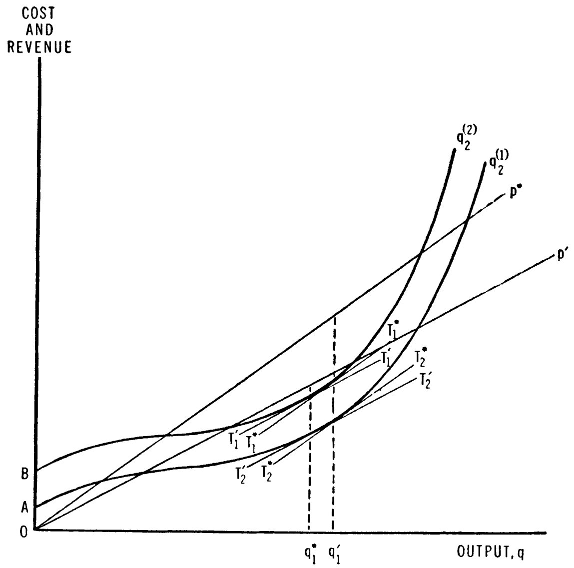

Fig. 1

We are now able to state precisely the major results obtained thus far:

Theorem I: The presence of separable externalities in a firm's cost function implies, under the usual convexity assumption, that the firm must follow the "marginal cost equals price" rule in order to

[13] This assumes, of course, that average variable costs are covered.

maximize profits. Conversely, an operational "marginal cost equals price" rule for profit maximization implies that any technological externality must be separable.[14]

Our previous discussion has dealt extensively with the first part of this theorem; we content ourselves with two further observations. First, note that as long as the firm remains in operation, the value of the externality is irrelevant to the firm's attempt to maximize profits because (*a*) by assumption the possibilities of merger, collusion, or co-operation are not admitted here and (*b*) the firm by definition cannot exercise control over the externality in the absence of these possibilities. Second, we note that by definition separability means, in this context, that marginal cost is defined in terms of the firm's own output variable. Consequently, if

$$\frac{\partial C_1}{\partial q_1} \neq p ,$$

profits are not maximal and are made so, by well-known theorems, only when this equality is established.

The second part of the theorem has not been explicitly considered. Again we choose to proceed along intuitively appealing routes rather than with nothing but mathematical rigor. Accordingly, we again turn our attention to Figure 1. We wish to show that, if the firm can follow operationally the "marginal cost equals price" rule, then the externality must be separable.

First, for the "price equals marginal cost" rule to be operational, the firm must be able to equate marginal cost to price over all relevant ranges of price variation. Accordingly, we have drawn two total revenue curves, Op' and Op^*, corresponding to the two prices which we wish explicitly to consider. Let us consider now the price represented by the slope of Op'. Call this price p'. If the firm follows the "marginal cost equals price" rule, then it must equate the tangent to its cost curve, say $Aq_2^{(1)}$, with price p'. Therefore, the firm would choose output q_1^* since the slope of $T_2'T_2'$ equals the slope of Op' at this point. But for the marginal cost curve to be uniquely defined, any other cost curve (caused by a change in the value of the externality), say $Bq_2^{(2)}$, must have a tangent $T_1'T_1'$ with the same slope at q_1^*. This means that the total cost curves differ from one another by some constant amount at q_1^*.

Now let us consider any other price, p^*, which is represented by the slope of Op^*. The same argument applies here as was used above. The firm must equate the slope of the tangent to its cost curve, say $T_2^*T_2^*$, to price p^*. Therefore, output q_1' is chosen. But in order for the marginal cost curve to be unambiguously defined,[15] the tangents to all other total cost curves —$T_1^*T_1^*$, for example—must have the same slope at q_1' as $T_2^*T_2^*$. Therefore, the total cost curves differ from each other by some constant amount at q_1'.

Since, as long as the firm remains in operation, the above argument applies for all prices, it follows that the total cost curves differ from each other only by some constant amount. This can only be the case if the externalities are separable.

Theorem II: Separable externalities imply the game theoretic concept of dominance.

[14] We use the term "operational" in a special sense here. Our usage requires that the firm be able to know its own marginal cost curve in the absence of knowledge of the other firm's output decisions.

[15] The terms "uniquely defined" and "unambiguously defined" are used in a special manner here. We mean that the marginal cost curve is defined if one firm does not know the output of the other firm

We have shown that separable externalities imply the "price equals marginal cost" rule for profit maximization. But this means, as long as the firm remains in operation, that the same output is optimal for the firm no matter what value the externality takes on. This means that one firm's output decision is independent of the other firm's decision and this is, by definition, the game-theoretic concept of dominance in the context of our approach.

Let us now turn our attention to discrete cases in order that we may explore the welfare implications of separable externalities with the analytical tools of game theory without having to resort to the mathematical complications of continuous games. It can be argued, of course, that most firms will not know their marginal cost functions exactly. But it is usually assumed that to a satisfactory degree of approximation they can at least determine noticeable stepped increments in cost for discrete variations in production. And if this argument is accepted, then the discrete formulation is entirely appropriate. We prefer the discrete formulation, however, for additional reasons. First, it entails no real loss of generality and avoids resort to more complicated mathematics. Second, the theory of discrete games is easily available and widely known.[16]

Before proceeding to the discrete game formulation, however, it is necessary to digress briefly on the meaning of the "price equals marginal cost" rule for discrete cases.

We first note that, under the usual convexity assumptions, the "marginal cost pricing" rule is derived from a maximization problem. This fact is the essence of our argument for the discrete case. Each firm is assumed to attempt to pick that (discrete) output for which its profits are maximal. What is required is that we formalize this condition—the analogue of the marginal-cost-pricing rule—for the discrete case.

Let us consider first the simpler problem where externalities are not present. Suppose that the cost function $C(q)$ of a firm is defined for only certain values of q. In other words, this firm produces only at discrete levels of output. We now define a set Q

$$Q = [q \mid C(q) \text{defined}] . \quad (11)$$

In other words, the set Q is composed of all the values of q which the firm could choose. Still using P to represent profits and p price, we note that the firm would desire to maximize

$$P(q) = pq - C(q) \quad (12)$$

over the set Q. Suppose that q^* is that feasible output for which (12) is maximal. Then it is obvious that

$$P(q^* + \Delta q) - P(q^*) \leq 0 \quad (13)$$

must hold for all admissible choices for Δq since otherwise q^* could not be the most profitable output as was assumed. Now we may use the profit function (12) in order to rewrite (13) in the following form:

$$\begin{aligned} p(q^* + \Delta q) - C(q^* + \Delta q) \\ - pq^* + C(q^*) \leq 0 . \end{aligned} \quad (14)$$

From (14) we can obtain

$$p\Delta q \leq C(q^* + \Delta q) - C(q^*) . \quad (15)$$

And from this the formal statement for the discrete analogue for the "marginal cost equals price" rule becomes rather obvious.

[16] For an especially clear and understandable exposition on discrete games, see R. Duncan Luce and Howard Raiffa, *Games and Decisions* (New York: John Wiley & Sons, 1957), esp. chap. v.

$$p \leq \frac{C(q^* + \Delta q) - C(q^*)}{\Delta q} \quad \text{for } \Delta q > 0$$
$$p \geq \frac{C(q^* + \Delta q) - C(q^*)}{\Delta q} \quad \text{for } \Delta q < 0. \qquad (16)$$

In other words, the optimal output q^* must satisfy the condition that if output were to be increased by any admissible amount Δq, then price would be less than the slope of the line segment joining the two points $C(q^* + \Delta q)$ and $C(q^*)$. Conversely, if output were to be decreased by any admissible amount Δq, then price would be greater than the slope of the line segment joining these two points.

Let us now examine this discrete problem when externalities are present. Suppose that the cost function $C(q_1, q_2)$ of the firm under consideration is separable, so that

$$C(q_1, q_2) = \bar{c}(q_1) + \hat{c}(q_2). \qquad (17)$$

Since this firm is assumed to produce only certain (discrete) outputs, $C(q_1, q_2)$ is defined only for these values of q_1. Let a set Q_1, composed of all values of q_1 which can be chosen, be defined as in (11). Note that the firm would desire to maximize a profit function

$$P(q_1, q_2) = pq_1 - C(q_1, q_2) \qquad (18)$$

over the set Q_1. Assuming that q_1^* is a feasible output for which (18) is maximal, given any particular value of q_2, then it follows that

$$P(q_1^* + \Delta q_1, q_2) - P(q_1^*, q_2) \leq 0 \qquad (19)$$

must hold for all admissible choices of Δq_1. As was done for the previous example, we may substitute the profit function into the above in order to obtain

$$p(q_1^* + \Delta q_1) - \bar{c}(q_1^* + \Delta q_1) - \hat{c}(q_2) - pq_1^* + \bar{c}(q_1^*) + \hat{c}(q_2) \leq 0. \qquad (20)$$

Collecting terms gives

$$p\Delta q_1 \leq \bar{c}(q_1^* + \Delta q_1) - \bar{c}(q_1^*), \qquad (21)$$

from which it follows that

$$p \leq \frac{\bar{c}(q_1^* + \Delta q_1) - \bar{c}(q_1^*)}{\Delta q_1} \quad \text{for } \Delta q_1 > 0$$
$$p \geq \frac{\bar{c}(q_1^* + \Delta q_1) - \bar{c}(q_1^*)}{\Delta q_1} \quad \text{for } \Delta q_1 < 0. \qquad (22)$$

These are, of course, the discrete analogue of the "marginal cost equals price" rule for the case with separable externalities, and, as was expected, there is a similarity between (22) and (16). Note that the externality, $\hat{c}(q_2)$, does not appear in (22). Evidently, in the case of separable externalities, discreteness does not affect the results which were obtained for the continuous case. The firm may calculate the cost associated with each discrete change in output and, therefore, it can determine its output unambiguously in accordance with the stipulated rule (22). In particular, it is interesting to note that no matter what value the externality takes on, the firm will still select, as long as it remains in operation, the output q_1^*.

It should be obvious that both Theorem I and Theorem II apply for the discrete case with appropriate modifications in wording. In other words, Theorem I can be restated as follows: the presence of separable externalities in a firm's discrete cost function implies that the firm must follow the discrete analogue (22) to the "marginal cost equals price" rule in order to maximize profits. Conversely, the discrete analogue to the "marginal cost equals price" rule for profit maximization implies that any technological externality must be separable. Theorem II is applicable as it stands. No rewording is needed.

Since the proofs for the discrete case follow the same form as those for the continuous case, we choose to omit them here. The theorems are intuitively obvious in any event.

We are now able to formulate this problem in terms of a discrete, two-person, non-zero sum game in order to explore the welfare implications of separable externalities. Thus let us represent the various combinations of outputs (strategies) of the two firms by the following game matrix:

$$\text{Firm 1} \quad \begin{matrix} i=1 \\ \cdot \\ \cdot \\ \cdot \\ n \end{matrix} \overset{\substack{\text{Firm 2} \\ j=1,\ldots,m}}{\begin{bmatrix} \\ (a_{ij}, b_{ij}) \\ \\ \end{bmatrix}} . \qquad (23)$$

The profit accruing to firm 1 when it chooses the output associated with row i, and firm 2 chooses the output associated with column j, is represented by a_{ij}. Similarly, b_{ij} represents the profit accruing to firm 2 when the indicated output choices are made.[17]

Consonant with the assumptions of classical analysis and with our assumptions of this section, it is assumed that the game is non-co-operative. Then, since we assume separability and a competitive market, Theorem I tells us that the supposition of profit maximization gives "price equals marginal cost" for each firm. But, since separability requires that the marginal cost of each firm be given entirely in terms of its own output, this rule means that for firm 1 there exists a row for which, given any particular output of firm 2, its profits are maximal. Similarly, for firm 2 there exists a column for which, given any particular output of firm 1, its profits are greater than at any other output level. These results are apparent from the discussion concerning Theorem I where it was shown that the value of the externality (or the output decision of the other firm) was irrelevant for the optimal output decision by either of the firms under consideration. Theorem II tells us that this is the Von Neumann-Morgenstern concept of dominance.

It is true, of course, that individual maxima need not—indeed, in general will not—be equal to the maxima for both firms considered together as a co-ordinated unit. This is apparent from (18) and was the cause of our concern in section 2 when we attempted to show that, under competition, joint maximization is a necessary condition for a welfare optimum. A simple example, attributed to A. W. Tucker and developed in an entirely different context, may be helpful here.[18]

Assume that the payoff (profit) matrix for the two "interrelated" firms is as follows:

$$\text{Firm 1} \begin{matrix} R_1 \\ R_2 \end{matrix} \overset{\substack{\text{Firm 2} \\ Q_1 \qquad\qquad Q_2}}{\begin{bmatrix} (0.9, 0.9) & (0, 1) \\ (1, 0) & (0.1, 0.1) \end{bmatrix}} \qquad (24)$$

Clearly, row R_2 is dominant for firm 1 since $1 > 0.9$ and $0.1 > 0$. Column Q_2 is dominant for firm 2 since $1 > 0.9$ and $0.1 > 0$. Hence, the non-co-operative solution is R_2, Q_2, which yields a profit of 0.1 to each firm.[19]

[17] It is assumed that $b_{ij}, a_{ij} \geq 0$ for all relevant i, j choices so that there is no problem of covering average variable cost.

[18] This example, known as the "Prisoner's Dilemma," is adopted from Luce and Raiffa, *op. cit.*, pp. 94–102.

[19] Some empirical data confirm the hypothesis that, if communication does not take place, players continually choose the *individual rational* strategy. In terms of the above example, this choice would mean R_2 and Q_2. For the results of the laboratory

Since a competitive market structure is assumed here, the appropriate welfare solution is the joint maximum R_1, Q_1, with an individual profit to each firm of 0.9. The problem of social policy is how to bring about this solution.

IV. SOME POSSIBLE APPROACHES

Let us now discuss possible ways in which the desired welfare solution could be accomplished. First, the government could adopt a "planning" approach and impose direct constraints so that the appropriate outputs—R_1, Q_1 for the previous example—would be chosen. While we shall postpone a detailed discussion of this "constrained-game" formulation until a later point in the paper, it is appropriate that we point out here some of the difficulties associated with this approach. In a more complicated situation than that represented by our simple example (24), such as the more general cases conceptualized in the matrix (23) or the continuous game formulation, the governmental policy-maker must possess some knowledge of the cost functions of the individual firms, or at least some knowledge of the entries in the payoff matrix, in order to accomplish this solution. Since many firms and multitudes of externalities may be present in the real world, the problem of gaining adequate information appears to be very great.[20]

Second, the welfare solution could be achieved by imposing a tax-subsidy arrangement that brought the appropriate output decisions—for example, R_1, Q_1 in (24)—into a position of dominance. Although it might appear a simple task for the policy-maker to accomplish this result in the simple example (24), since an exact knowledge of the costs and profits of each firm would not be required, but only sufficiently large tax-subsidy arrangements to reverse the indicated dominance, the general cases are more complicated and demanding. Therefore, it seems entirely appropriate to consider the conditions which the tax or subsidy would have to meet in order to accomplish the desired solution.

Let us consider, first, the continuous case. The policy-maker must be able to determine (at least approximately) appropriate outputs for each firm in order that proper taxes and subsidies can be levied. This could be accomplished in our two-firm examples by solving equations (4) simultaneously in order to obtain the q_1 and q_2 that achieve joint maximization. Designate these welfare-optimal outputs q_1^* and q_2^*. Then, using t to represent both taxes and subsidies (a positive t indicates a subsidy and a negative t a tax), the proper subvention or penalty would be given by

$$p+t_1=\frac{\partial C_1(q_1,q_2)}{\partial q_1}\bigg|\,q_1^*$$
$$p+t_2=\frac{\partial C_2(q_1,q_2)}{\partial q_2}\bigg|\,q_2^*\,. \qquad (25)$$

Of course, p is easily available. But the partial derivatives which have to be evaluated here (and available for a solution to [4]) may not be so readily obtainable. To say the least, the policy-maker would have to make an intensive study in order to obtain the desired information.

We now turn our attention to the discrete case. Here, too, the policy-maker must be able to determine appropriate

experiments, see A. Scodel, J. S. Minas, P. Ratoosh, and M. Lipetz, "Some Descriptive Aspects of Two-Person Non-Zero Sum Games," *Journal of Conflict Resolution*, III (1959), 114–19.

[20] Throughout this paper we assume that the government desires to maximize welfare, an assumption which may not always be completely appropriate.

welfare outputs—perhaps by simultaneously solving discrete equations for a joint maxima—in order to determine proper taxes and subsidies. Designate a welfare optimal output for firm 1 by q_1^*.[21] Since the firm is free to choose any output q_1 which it pleases, it is obvious that, if the welfare optimal output q_1^* is to be chosen, the per unit tax or subsidy must be such that (19) is satisfied. If t is used as it was in the previous discussion, then $p + t_1$ can be used instead of p in the derivation (equations [20] and [21]) of condition (22) with the result that

$$p + t_1 \leq \frac{\bar{c}(q_1^* + \Delta q_1) - \bar{c}(q_1^*)}{\Delta q_1} \quad \text{for } \Delta q_1 > 0$$

$$p + t_1 \geq \frac{\bar{c}(q_1^* + \Delta q_1) - \bar{c}(q_1^*)}{\Delta q_1} \quad \text{for } \Delta q_1 < 0 \qquad (26)$$

must obtain. Once again, p is easily available for the policy-maker, but the slopes of the line segments joining $\bar{c}(q_1^* + \Delta q_1)$ and $\bar{c}(q_1^*)$ may not be so readily obtainable although, of course, an evaluation may be possible. The interesting point concerning the discrete case, however, is that taxes and subsidies are not necessarily determined uniquely. Instead, there may be limits, which depend upon the relative slopes of the line segments, between which the taxes and subsidies can vary.[22]

It is almost trivial to point out, however, that in both the discrete and continuous cases the tax-subsidy solution does not possess one of the most important characteristics of a perfectly functioning market mechanism. Each time there is a technological change which affects the firms under consideration, there would have to be a recomputation and adjustment of the taxes and subsidies. A perfectly functioning market, on the other hand, automatically adjusts for these changes (at least from the point of view of comparative statics).

As a final policy approach, we note that, provided that the market structure can remain competitive, forces may exist within the price system that will tend to produce the optimal solution even with no action by the government. For merger might be mutually beneficial to both firms and could be expected to occur if either the rules of society or possible internal problems of decentralized administration within the merged entity did not prevent such action.[23]

It is interesting to note that Meade,

21 We use "a welfare optimal output" here because with discreteness it is likely that more than one optimum output exists. Also note that we use q_1^* (and q_2^*) here and in the previous discussion of the continuous case as welfare-optimal outputs which are determined by the policy-maker. Since these become optimal outputs for the individual firms *only after* proper taxes and subsidies have been levied, they are not to be confused with the firm maximal outputs of the previous section which were designated by the same symbols.

22 Many of our externality problems appear to fall into the discrete case. For example, problems associated with plant location, municipal zoning, or even minimum building codes can be viewed as discrete. (see Tjalling C. Koopmans and Martin Beckmann, "Assignment Problems and the Location of Economic Activities," *Econometrica*, XXXIV [January, 1957], 53–76; and Otto A. Davis and Andrew B. Whinston, "The Economics of Complex Systems: The Case of Municipal Zoning" [O.N.R. Research Memorandum, Graduate School of Industrial Administration, Carnegie Institute of Technology]). Unfortunately, it appears as if many of these externalities are not separable.

23 A further problem for research would be to determine the "fair" terms under which the merger would take place, with reference to the division of the gains among the individual stockholders in the two firms. While this is not the appropriate place to go into this problem, it appears that a possible method of analysis would be along the lines outlined by L. S. Shapley, "A Value for *n*-Person Games," *Annals of Mathematics*, Study No. 28 (Princeton, N.J.: Princeton University Press).

who produced the modern restatement of the classical tax-subsidy prescription for the externality problem, does not even consider the possibility of the merger solution in his "unpaid factors case."[24] In fact, in the particular example that Meade uses, this solution appears possible. The example involves apple growers and honey producers. The nectar from apple blossoms is a scarce commodity which, it is postulated, cannot be priced. (Later in his analysis Meade also assumes that the bees help to pollinate the apples, thus creating a mutual externality.) Thus Meade advocates the classical solution of taxes and subsidies. If, however, there exists a spatial distribution of apple growers and honey producers such that after merger one firm's bees would not be expected to wander over into some other firm's apple orchard, the externality would be internalized and, if competition could be maintained, the optimal solution would result.

V. NON-SEPARABLE COST FUNCTIONS

Let us now consider the non-separable type of externality where it is not clear that the usual solution of taxes and subsidies will work, even at the conceptual level. The difference between the separable and the non-separable cases lies in the fact that externality enters the cost function in a "multiplicative" manner rather than in a manner which is strictly "additive." In other words, the separability condition (7) is not satisfied.

Once again, it is necessary for us to discuss both continuous and discrete cases. We consider the continuous case first. And it also seems completely appropriate, because of the greater clarity achieved, to proceed by assuming specific cost functions:

$$\begin{aligned} C_1(q_1, q_2) &= A_1 q_1^n + B_1 q_1 q_2^m \\ C_2(q_1, q_2) &= A_2 q_2^r - B_2 q_2^t q_1^s . \end{aligned} \qquad (27)$$

Profit maximization by each individual firm implies the following relationships:

$$\begin{aligned} p &= \frac{\partial C_1}{\partial q_1} = nA_1 q_1^{n-1} + B_1 q_2^m \\ p &= \frac{\partial C_2}{\partial q_2} = rA_2 q_2^{r-1} - tB_2 q_2^{t-1} q_1^s . \end{aligned} \qquad (28)$$

Now note that, from the individual firm's standpoint, marginal cost is defined not only in terms of the variable which it can control—its own output—but also in terms of the variable which it cannot control—the other firm's output. Both q_1 and q_2 enter into each equation. How, then, can the firm choose an output which will maximize its profit when its own marginal cost depends upon the decision of the other firm?

Let us now compare and contrast the continuous cases of separable and non-separable externalities in order to bring out the effects of non-separability. It will be recalled from Figure 1 that a change in the value of a separable externality had the effect of vertically shifting the total cost curve upward or downward. The curves differ from one another only by the value of a constant. Thus at any given output the slopes of the tangent curves—that is, marginal cost—were not affected by alterations in the value of the externality. The non-separable case does not have this property. While alterations in the value of the externality cause the total cost curve to shift upward and downward, there is no reason to expect that this shift will be a simple vertical displacement. In general, total cost curves generated by changes in the value of a non-separable externality will not simply differ from each other by the value of a constant. Rotation or some

[24] Meade, *op. cit.*, pp. 56–61.

other type of alteration is likely to take place when the externality changes in value. This fact is, of course, obvious from the fact that the separability condition (7) is not satisfied. It can also be seen from observation of the marginal cost curves (28) of our special example.

It is now obvious that the "marginal cost equals price" rule is affected by non-separable externalities. For whereas separable externalities had a strictly intramarginal effect, non-separable externalities affect the margin. In (28) the externality enters into the definition of marginal cost. Since by definition the firm cannot control the value of the externality, it clearly follows that the firm will find it difficult *operationally* to follow the "marginal cost equals price" rule of profit maximization. The fact is that, for the non-separable case, marginal cost is ambiguously defined in terms of the firm's own output.

From the game-theoretic standpoint, this type of externality suggests the absence of dominance. This point, too, is obvious from the separability condition (7) and from the marginal cost curves (28) of our example. For, supposing that the firm desires to maximize profits, it must alter its output with every change in the value of the externality in order to attempt to equate marginal costs with price. This means that the optimal output (strategy) of one firm depends upon the output (strategy) selected by the other firm. Such interdependence is the essence of non-dominance.

Now let us examine the discrete case, in order to utilize the analytical tools of the theory of games in exploring the welfare implications of non-separable externalities. It is necessary, of course, that we state the discrete analogue for the "marginal cost equals price" rule for the non-separable case. Then, assuming that $C(q_1, q_2)$ is a non-separable cost function which is defined for a set Q_1 of discrete outputs, we may derive

$$
\begin{aligned}
p &\leq \frac{C(q_1^* + \Delta q_1, q_2) - C(q_1^*, q_2)}{\Delta q_1} \qquad \text{for } \Delta q_1 > 0 \\
p &\geq \frac{C(q_1^* + \Delta q_1, q_2) - C(q_1^*, q_2)}{\Delta q_1} \qquad \text{for } \Delta q_1 < 0
\end{aligned}
\tag{29}
$$

as the desired rule.[25] But note that, since $C(q_1, q_2)$ is non-separable, the terms involving the externality cannot be canceled out and that condition (29), unlike its separable counterpart (22), involves q_2. Thus q_2 affects the discrete analogue to marginal cost. This means that the output q_1^* which satisfies (29) must depend, in general, upon the value which q_2 assumes. Therefore, in the discrete as in the continuous case, non-separable ex-

[25] We present this derivation in a footnote since it is similar to that used for the separable case. The firm desires to maximize the profit function

$$P(q_1, q_2) = pq_1 - C(q_1, q_2) \tag{a}$$

over the set Q_1 of possible outputs. By assumption the firm cannot control the value of the externality q_2. But for some given q_2 it is obvious that any profit maximizing output q_1^* must satisfy

$$P(q_1^* + \Delta q_1, q_2) - P(q_1^*, q_2) \leq 0 \tag{b}$$

for all admissible choices Δq_1. Substituting the actual profit function (*a*) into (*b*) we obtain

$$p(q_1^* + \Delta q_1) - C(q_1^* + \Delta q_1, q_2) - pq_1^* + C(q_1^*, q_2) \leq 0 \, . \tag{c}$$

Since $C(q_1, q_2)$ is non-separable, collecting terms gives

$$p\Delta q_1 \leq C(q_1^* + \Delta q_1, q_2) - C(q_1^*, q_2) \tag{d}$$

from which (29) follows directly.

ternalities introduce an interdependence in decision-making.

From the standpoint of discrete games, the presence of non-separable externalities suggests that there is no row-column dominance. In other words, in a matrix representation such as (23), for firm 1 there does *not* exist a row in which, for *every output* of firm 2, its profits are maximal. Similarly, for firm 2 there does *not* exist a column in which, for *every output* of firm 1, its profits are greater than at any other output level.[26] Non-dominance is evident, of course, from the fact that the externality enters the discrete analogue to marginal cost.

It seems clear that in both the continuous and discrete cases non-separable externalities introduce an interdependence between decision-making units. We have here, even in what is usually considered the certain world of competitive price theory, an example in which decisions must be made under uncertainty. It is this aspect of the externality problem which is roughly analogous to duopoly theory.[27] How can a firm determine its profit-maximizing output in this situation? One possible approach would be for each firm to attach subjective probability to its set of possible outputs and select that output which would maximize its expected profits.[28] A max-min approach might be another possibility. Or one can make various other assumptions concerning how the firms might act and react. But there seems to be *no a priori method* for determining the outputs (strategies) selected. Non-separable externalities raise the possibility of the non-existence of equilibrium.

The important point here, as in the separable case, is that there is no reason to expect the output which maximizes social benefit (meaning the solution which maximizes joint profits in the assumed competitive market) to be chosen. Again it seems desirable from the standpoint of society that either the game be constrained, the scores altered, or other changes be affected so that the appropriate welfare solution will emerge. But whereas the separable case raised only the problem of the misallocation of resources, the non-separable case raises both the problem of the misallocation of resources and the problem of mal-coordination of decision-making because of the interdependence between marginal cost curves.

VI. POLICY APPROACHES AND EQUILIBRIUM

We now discuss possible ways in which the welfare objective might be accomplished in the case of non-separable

[26] A very simple example of non-dominance is the following:

$$\text{Firm 1}\ \begin{matrix} R_1 \\ R_2 \end{matrix} \overset{\text{Firm 2}}{\overset{\begin{matrix} Q_1 & \quad Q_2 \end{matrix}}{\begin{bmatrix} (.9, 0) & (.1, .9) \\ (0, 1) & (1, .1) \end{bmatrix}}}$$

It is not clear whether firm 1 will choose strategy (output) R_1 where $.9 > 0$ or strategy R_2 where $1 > .1$. Similarly, it is not clear whether firm 2 will choose strategy (output) Q_2 where $.9 > 0$ or Q_1 where $1 > .1$.

[27] It is to be emphasized, however, that this resemblance to duopoly theory is somewhat superficial. Although we have developed our analysis in terms of two firms, this approach has been used only for expository convenience. It is obvious that, with non-separable externalities, interdependence between decision-making units is the source of the difficulty, and this interdependence exists if the number of firms is two, three, or n, where n can be an indeterminately large number.

[28] Such an approach would be similar to that of L. J. Savage, *The Foundations of Statistics* (New York: John Wiley & Sons, 1954). On the other hand, Von Neumann and Morgenstern assert that the difficulties inherent in situations in which neither participant controls the relevant variables cannot be obviated by recourse to statistical assumptions and analysis (*The Theory of Games and Economic Behavior* [Princeton, N.J.: Princeton University Press, 1947], p. 11).

externalities. The first possibility is the proposed merger solution. With no action by the government and provided that the market can remain competitive, forces may exist within the price system which will tend to produce the optimal welfare solution, since merger might be mutually beneficial to firms operating under the postulated types of externalities. It merits repeating here that, if problems such as might be associated with decentralized administration do not prevent merger, firms are motivated to merge until the postulated externalities which can be "internalized" are eliminated; that is, until the "natural unit" for decision-making is reached.[29]

Second, the government could try the classic prescription of levying special taxes and subsidies. This solution, however, appears even less feasible in this case, and this point can be seen clearly by a comparison with the separable externality example. In the latter case a dominant solution existed. The governmental policy-maker, if he knew the relevant cost functions, could levy excise taxes and give subsidies on output as a constant function for each firm according to rules (25) and (26) so that profit maximizing firms would be induced to choose the optimal welfare outputs. However, in the non-separable externality case, even assuming that the governmental policy-maker knows the relevant cost functions and desires to maximize welfare, there seems to be no dominant solution to aim at. It is well known that there does not exist a known, unique equilibrium solution in pure strategies for this type of game (which is not to say that the firms will not make a decision on each play, but only that the decisions cannot be predicted). Thus it seems improbable that the governmental policy-maker would know the strategies which the individual firms would play since, as was pointed out above, there is no a priori method for determining the outputs which might be selected. In fact, for the policy-maker to be able to determine the strategy which individual firms might be playing, it would seem necessary, in the absence of a priori methods, to obtain information concerning the psychologies of the managers, their "taste" for risk, and so on. Of course, this knowledge is not readily available; and if the policy-maker did not know these strategies, he could not possibly predict the reaction of the firms if he tried to rotate the total revenue curve (or, what is analytically the same, shift the price line; shift the marginal cost functions; or, in game-theoretic terms, alter the payoffs) through the tax-subsidy method. Thus, even if the policy-maker determined what might be considered "appropriate" subventions or penalties by methods analogous to those suggested by (25) and (26) for the separable case, there is no assurance—and even little likelihood—that the firms would voluntarily choose the welfare-optimal outputs. Non-separable externalities affect firms' marginal costs and thus create interaction between the decision-making efforts of individual firms.[30]

[29] The notion of a natural unit can be taken as roughly corresponding to the concept of a stable coalition in n-person game theory. In this respect it might be noted that throughout this analysis we have chosen to ignore the possible instabilities which might be associated with entry into the industry.

[30] It might be suggested that the tax-subsidy scheme could be made appropriate even in this case by having the policy-maker simply solve for the optimal outputs and offer a subsidy conditional upon firms producing those specified outputs. But this would entail abandoning the advantages of a decentralized market system, since the policy-maker must actually specify acceptable outputs. Also note that if acceptable outputs are specified by policy, then there would seem to be little reason to

It follows from the above analysis that the classical tax-subsidy solution, originally stated by Marshall and Pigou and recently restated by Meade, breaks down for the case of this non-separable type of externality.[31]

Meade's analysis, although it is carefully developed and illuminating, is especially interesting in this respect. Like most of the other writers on this subject, he uses for much of his analysis a general functional notation which makes it impossible to determine accurately whether the externalities are separable or non-separable.[32] But, perhaps because he deals more thoroughly with the problem than other writers, Meade introduces in his "Atmosphere Case"[33] the production functions

$$x_1 = H_1(l_1, c_1)A_1(x_2)$$

$$x_2 = H_2(l_2, c_2)A_2(x_1),$$

which necessarily contain externalities of the non-separable type because of the "multiplicative" terms $A_1(x_2)$ and $A_2(x_1)$. It is true, of course, that Meade is dealing with industries rather than firms and that he (at least implicitly) assumes that the firms in each of these industries takes the output of the other industry as a parametric constant. On the basis of this assumption it might be argued that a tax-subsidy scheme could work in principle, although it would require elaborate computations which would have to be repeated each time a price alteration, or a technological change occurred. However, the major objective of our game-theoretic analysis of non-separable externalities has been to show that this type of interdependence creates uncertainties which, in turn, make such an assumption arbitrary and unwarranted. And our analysis, which has been developed for two firms but which certainly can be extended to cover any number of firms, shows that in the non-dominance case a stable equilibrium is unlikely to be achieved. There seems to be no a priori method for determining what strategies (output choices) firms will select in the presence of non-separable externalities.

Since our discussion of the non-dominance case has been based upon a game-theoretical analysis, an approach which we feel is to be preferred to the approach used by Meade and others, it might be argued that our conclusions that there is no *obvious* equilibrium solution in pure strategies are based solely upon our choice of tools of analysis and that the "usual" tools do logically show that an equilibrium must be achieved. The usual tools involve the solution of a set of simultaneous equations. We shall show that an analysis with these usual tools need not imply the existence of an

offer conditional subsidies. Why not simply dictate to the firms that certain outputs must be produced? This, however, weakens the case for private ownership of the facilities under consideration.

[31] Our results here hold for the case of reciprocal, non-separable externalities. If the externalities are not reciprocal in any sense—that is, if firm 1 imposes a non-separable externality upon firm 2 but not vice versa, and if there are no other externality-creating firms which are relevant—then our analysis does not hold. For a given output of firm 1, firm 2's marginal cost curve will be precisely determined so that, at least conceptually, the policy-maker can compute the appropriate rate of the tax or subsidy; but, of course, this rate will have to be recomputed each time firm 1 alters its output. If the situation is more complex (as for example, firm 1 imposing a non-separable externality upon firm 2, firm 2 imposing one upon firm 3, and firm 3 upon firm 1), then our analysis does hold. It is the necessity for "simultaneous" decision-making in the presence of this type of interdependence that creates the uncertainty and difficulty here.

[32] Meade makes the usual assumption of linear homogeneous production functions. It can be shown, however, that this assumption does not rule out the possibility of non-separable externalities.

[33] Meade, *op. cit.*, pp. 61–66.

equilibrium solution. Our main point will be that the assumptions needed for an equilibrium solution are not consistent with the other assumptions of the model.

Let us consider the simple example upon which we based much of our analysis of the non-separable case. The usual method would be to solve the following equations:

$$\frac{\partial P_1}{\partial q_1} = p - nA_1 q_1^{n-1} - B_1 q_2^m = 0$$
$$\frac{\partial P_2}{\partial q_2} = p - rA_2 q_2^{r-1} + tB_2 q_2^{t-1} q_1^s = 0 . \tag{30}$$

Let q_1^* and q_2^* be the "equilibrium" solutions to these two equations: if both firms happened to select the outputs q_1^* and q_2^*, neither firm would have any desire to change its output plans upon hearing of the output plans of the other firm. We assume that these solutions are unique and non-negative. Suppose that, for some reason, some other outputs q_1' and q_2' were chosen. Then each firm would desire to alter its output in order to adjust for the externality in its efforts to maximize profits. But when the new outputs were chosen, say q_1'' and q_2'', $q_1'' \neq q_1^*$, $q_2'' \neq q_2^*$, then it would be desirable to alter outputs again and so on ad infinitum.

We now distinguish two sets of assumptions that might lead one to infer that the process of "an infinite progression" of mutual adjustments and readjustments would lead to an "equilibrium" solution.

First, the firms might be assumed to communicate with each other, announcing tentative outputs but not producing until the "equilibrium" outputs q_1^* and q_2^* are announced. But if the firms are allowed to communicate, it seems unlikely that they would just exchange output data. Why would they not also exchange data about cost functions? But if they exchange data about cost functions (or even if they just communicate), why would they not make the most of the ability to communicate by exchanging information that would lead to a *joint maximization*, since both would stand to gain thereby? Thus, on the basis of the simple communication assumption it would seem that they would exchange such information as would make them act *as if* they were merged instead of trying to seek the so-called "equilibrium" solution.

Under the other set of assumptions, the firms strive to reach "equilibrium outputs in the long run." Since one firm cannot be assumed to know the other firm's cost function, there is little reason to suspect that in the initial period the "equilibrium" outputs would be chosen. Each period each firm observes the other's behavior, which it desires to take into account in its own decisions. Thus each firm would be led to try to predict, on the basis of past data, the other firm's output for the next period. The firms observe, predict, make their decisions accordingly, and produce. This process goes on period after period. As we saw earlier, unless the equilibrium values are chosen, both outputs will change. A (somewhat naïve) application of Muth's "Rational Expectations Hypothesis"[34] suggests that the one hypothesis that the firms would *not* be expected to use would be that the other firm would not change its output in the next period. Change would be expected. Thus, under this hypothesis, even if the two firms do happen to reach the unique "equilibrium" outputs after "an infinite progression" of periods of adjustment, each

[34] John F. Muth, "Rational Expectations and the Theory of Price Movements," *Econometrica*, XXIX, No. 3 (July, 1961), 315–35.

firm would expect the other to alter output in the following period; this expectation would be taken into account in making decisions for that next period, and equilibrium would not be maintained. For it is only at one unique point that the equations in the model predict no change for the next period, and this point is unknown to the firms.

It follows from the above line of reasoning that the only way in which the firms would reach and maintain the equilibrium solution would be by one firm assuming that the other firm would not change its output from period to period even though observations revealed otherwise.

However, the game-theoretic formulation shows clearly that there is unlikely to be a unique equilibrium solution (the case of non-dominance) for this type of externality.[35] In order to know how the firms would react to the possible taxes and subsidies, the policy-maker would have to know, as has been pointed out, not only the exact nature of the cost functions, but also the psychological characteristics of the decision-makers within the firm or at least the general qualitative properties of their preference-satisfaction functions to be able to predict the strategies which might be chosen by the players.

Finally, the governmental policy-maker might adopt the method of the constrained game approach. While this approach is subject to the difficulties of gaining adequate information that were pointed out previously, it does seem to offer hope for some type of solution either where merger will not work because of the impossibility of "internalizing" the externalities, or where it will result in the creation of monopoly.

The government could try, of course, to "strictly constrain" the game by dictating appropriate outputs to the firms. However, granted our ethical bias against such direct planning, we assume that decentralized (non-governmental) decision-making is desirable wherever it is possible. Thus we propose to discuss now some cases where it may be possible to employ a combination of various constraints and the pricing mechanism in order to obtain an approximation to the appropriate welfare solution.

Let us consider a simple case where there is "almost" row-column dominance:

Firm 2

Firm 1	Q_1	Q_2	Q_3	Q_4
R_1	(10, 10)	(4, 17)	(2, 12)	(6, 4)
R_2	(12, 3)	(15, 8)	(3, 6)	(8, 9)
R_3	(3, 4)	(4, 15)	(2, 10)	(9, 7)
R_4	(13, 8)	(7, 3)	(5, 14)	(11, 11)

(31)

Row R_2 dominates row R_1 and column Q_3 dominates column Q_1. The remaining rows and columns introduce a non-dominance situation. If the governmen-

[35] The simultaneous solutions q_1^*, q_2^* to the set of equations (30) correspond to a Nash equilibrium point in a two-person, non-zero-sum game. Since we do not assume that one player knows the payoffs to the other player, expectations are all-important for the attainment of an equilibrium. In this respect, a comment by Luce and Raiffa on the Nash equilibrium seems especially relevant here. "These strategies will be in equilibrium provided that no player finds it is to his advantage to change to a different strategy *so long as he believes that the other players will not change*" (*op. cit.*, pp. 170–71). (Our italics.) For a discussion of further difficulties associated with this type of equilibrium notion see *ibid.*, pp. 171–73.

tal policy-maker imposes regulations so as to constrain behavior to rows R_1 and R_2 and columns Q_1 and Q_3, then the row-column dominance case is achieved. Once choices are so constrained, the use of the price mechanism will result in strategies R_2, Q_3 being chosen. It would now be possible to use taxes and subsidies in order to cause the "optimal" strategies R_1, Q_1 to be achieved.[36] In other examples, of course, the simple imposition of the direct constraints might result in an optimal, dominant solution.

The phenomenon of municipal zoning seems to afford a practical example of a case where "partial constraints" are used. The existence of height, area, and use restrictions may be viewed as evidence of the fact that these property features impose externalities upon certain types of other property.[37] As a very simple illustration, let us consider two entrepreneurs who own adjacent lots and who are trying to decide what types of plant they should erect upon their lots. Assume that for some reasons (noise, smoke, vibrations) the payoff to each entrepreneur depends upon the decision of the other entrepreneur. The game situation is evident. Assume, as seems reasonable in this instance, that the expected payoff associated with the operation of each type of plant at its individually optimal output level is similar to that represented in (31), where the subscripts now refer to the two entrepreneurs; but here entrepreneurs, instead of firms, make choices and the R's and the Q's represent types of plants rather than output levels. Obviously, from the viewpoint of social policy it is desirable that this game be constrained, and a method actually used in modern municipal zoning is to place use restrictions upon an area so that certain property uses (plants in this example) are excluded. In this illustration, uses R_3, R_4, Q_2, and Q_4 would certainly be forbidden in the area by any rational zoning ordinance. Granted these restrictions, the price mechanism would be allowed to operate, and each entrepreneur would be able to pick the most profitable type of plant not excluded by the restrictions. Of course, in this particular example, the unhappy choice R_2, Q_3 would result; so regulations might exclude these possibilities also. In other examples the simple elimination of rows and columns which cause non-dominance might produce a more desirable result; but this particular example does have the merit of suggesting the possibility of using taxes and subsidies in order to rely more on the pricing mechanism and less on direct constraints in municipal zoning.[38]

[36] The R_1, Q_1 solution is not the over-all optimum, which is, of course, R_2, Q_2. This latter solution, however, would be impossible to achieve unless there was "complete" regulation.

[37] See Otto A. Davis, "The Economics of Municipal Zoning" (unpublished Ph.D. dissertation, University of Virginia, 1959), for a more complete analysis of the zoning phenomenon which is not in game theoretic terms. See also Davis and Whinston, *op. cit.*, for a discussion which utilizes game theory and programming. It also might be pointed out that the phenomenon of urban renewal involves externalities and that a "preventative" solution can be obtained throughout a constrained game approach (see Davis and Whinston, "The Economics of Urban Renewal," *Law and Contemporary Problems*, Vol. XXVI [Winter, 1961]).

[38] Koopmans and Beckmann have shown, in a theoretical analysis of plant location, that in the absence of interaction a pricing mechanism will be given an optimal solution; but if interdependencies via transportation costs on "intermediate commodities" exist, no optimal equilibrium solution can be expected (see Tjalling C. Koopmans and Martin Beckmann, "Assignment Problems and the Location of Economic Activities," *Econometrica*, XXIV [January, 1957], 53–76).

VII. CONCLUDING REMARKS

Throughout much of the analysis of this paper we have assumed that the market structure remains competitive even after a sufficient number of mergers have occurred for the natural unit for decision-making to be achieved. This assumption has been necessary, of course, to permit welfare statements to be made. We do not propose to debate here the question of whether or not markets in the "real world" are competitive. Yet, since it might be inferred that a logical implication of our argument is that the natural unit for decision-making will be so large that the market structure will change, it must be pointed out that this implication is not necessarily true. The importance and extent (or "scope") of externalities is an empirical fact of the real world, and we have as yet no systematic evidence about this. A priori, it does seem plausible that in some instances the natural unit for decision-making will be so large that a competitive market structure will not exist, but this is not a novel conclusion. It has long been recognized that natural monopolies exist, and it is equally well known that some competitive markets exist.

The main point of our argument has been that the "classical" tax-subsidy solution to the problem of externalities on the production side would be difficult to achieve in the dominance case and impossible in the non-dominance case even if the government could be assumed to be trying to maximize welfare. We have argued that a much easier solution may exist in the case of competitive markets (and it is in this context that the problem has been analyzed by Meade and others) simply by allowing mergers to take place until the "natural unit" for decision-making has been achieved. Of course, mergers may not always be technically best. An implication of our argument is that it is less likely that the government will need to be concerned about externalities on the production side than is often thought, as long as the market is and remains competitive.

The difficult problem for the policy-maker when there are externalities on the production side arises when the market is not competitive, or when possible mergers which would "truly internalize" the externalities would result in a change in the market structure. Here some measures must be devised to indicate the possible welfare gains from merger and the welfare loss that would result from the divergence from the competitive situation.

Let us now consider briefly some possible methods of estimating the effects of externalities. Admittedly, the problems here are very difficult, and this is precisely the point which we have tried to emphasize in our previous discussion of possible solutions. It is not easy for the governmental policy-maker to obtain needed information on the nature of the cost functions and thus the entries in the payoff matrix. But, presumably, after study of each particular instance of externalities, some estimates could be made. So let us assume discreteness and consider the problems the policy-maker would face in the case of separable externalities when a possible merger which would truly internalize external effects might change the market structure from competitive to non-competitive. The policy-maker could use the estimated payoff matrix to determine the difference in total profits between the dominant solution and the maximum-profit solution. This difference in profits could be used as a crude measure of the change in welfare which is associated with a change from a situation of competitive markets and externalities to a situation of competitive markets and no externalities. Assuming merger is feasible, one must subtract from this difference a sum which would reflect the welfare loss associated with the alteration of the market to a situation of no externalities and a non-competitive market structure. One approach toward the estimation of this latter magnitude might be to consider this part of the welfare change as some function of the change in output and price that would result from the change in market structure.[39]

The policy-maker must also compare with this estimated net gain the net welfare gains that might result from direct regulation, inequality constraints, and other alternatives. These gains could be estimated by deducting from the estimate of the welfare gain calculated from the payoff matrix an estimate of the costs of the constraints themselves.

With an externality of the non-separable type the measurement problem is even more difficult. Cruder methods of approximation are necessary. Again assuming discreteness, statistical analysis of variance suggests one such possibility. Since, if there are no externalities present, the payoff matrix is composed of constant elements, a variance analysis would give a zero value here. Thus a variance analysis of the payoff matrix which gave a non-zero value could be taken as an approximation to the welfare value of the change from a situation of externalities and competition to a situation of no externalities and a competitive market. Thus the policy-maker might be able to make the appropriate comparisons as in the previous case. In situations where externalities exist and the market structure is non-competitive to begin with, the measurement problem is even more difficult. Yet, our a priori judgment is that this may be the more important area for policy choice by a welfare-maximizing government.

This paper has been limited largely to externalities on the production side. Other important externality problems associated with, for example, interrelated utility functions have not been treated here, although the game-theoretic approach does seem promising for future research in these areas.

[39] Franco Modigliani has suggested methods for predicting the change in output and price which accompany change in market structure ("New Developments on the Oligopoly Front," *Journal of Political Economy*, LXVI [June, 1958], 215–32).